A COMMON FIXED POINT THEOREM FOR THREE SELF MAPS

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Abstract:

A fixed point theorem of Sharma and yuel is extended to three self maps through the notion of compatible self-maps and associated sequence.

Keywords: Compatible self-maps associated sequence, orbitally complete metric space and Common fixed point.

Introduction:

Let (X, d) be a metric space. Self-maps P and Q are said to be commuting if $PQx = QPx$ for all $x \in X$.

Definition 1.1: According to Jungck [1], self-maps P and Q on X are compatible if $\lim_{n\to\infty} d(PQx_n, QPx_n) = 0$, when ever $(x_n)_{n=1}^{\infty}$ is a sequence in X such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qx_n = z$ for some $z \in X$.
We need the following definitions from [2].

Definition 1.2: Given $x_0 \in X$ and P, Q and R be self maps on X, we can find points $x_1, x_2, \dots, x_n, \dots$, then the associated sequence $(y_n)_{n=1}^{\infty}$ with the choice
$y_{2n+1} = Px_{2n+1} = Qx_{2n}\ ,\ y_{2n} = Px_{2n} = Rx_{2n-1}$ *for all* $n = 1,2,3, \dots \dots$ **(1)**
is called an (Q, R, P)-orbit or simply an orbit $O(x_0)$ at x_0.

Definition 1.3: The space X is (Q, R, P)-orbitally complete at x_0 if every Cauchy sequence in some orbit $O(x_0)$ converges in X.
Sharma et al [3] proved the following result:

Theorem A: *Let R be a continuous self-map on X such that*

$$d(Rx, Ry) \le \alpha d(x, y) + \frac{\beta d(x, Rx) d(y, Ry)}{d(x, y)} + \gamma[d(x, Rx) + d(y, Ry)]$$

$$+\delta[d(x,Ry)+d(y,Rx)] \qquad \text{for all } x,y \in X,\; x \neq y \qquad \textbf{... (2)}$$

Where α,β,γ *and* $\delta \geq 0$ *such that* $\alpha+\beta+2\gamma+2\delta<1$. *Then* R *has a unique fixed point.*

In this paper we extend Theorem A, to three self-maps using the notion of compatible self maps and associated sequence.

Main Result:

Theorem B. *Let* Q,R and P be continuous self maps on X *satisfying the inequality*

$$d(Qx,Ry) \leq \alpha d(Px,Py) + \frac{\beta d(Px,Qx)d(Py,Ry)}{d(Px,Py)} + \gamma[d(Px,Qx)+d(Py,Ry)]$$

$$+\delta[d(Px,Ry)+d(Py,Qx)] \qquad \text{for all } x,y \in X,\; Px \neq Py \quad \textbf{...(3)}$$

Where α,β,γ *and* $\delta \geq 0$. *Such that* $\alpha+\beta+2\gamma+2\delta<1$.

Suppose that

- *(i)* *There is an associated sequence with the choice* (1)
- *(ii)* X *is orbitally complete*
- *(iii)* (Q,P) *and* (R,P) *are compatible*

Then Q,R and P *have a unique common fixed point.*

Proof: From the inequality (3) and the definition of choice (1) we have

$$d(Px_{2n+1},Px_{2n}) = d(Qx_{2n},Rx_{2n-1})$$

$$\leq \alpha d(Px_{2n},Px_{2n-1}) + \frac{\beta d(Px_{2n},Qx_{2n})d(Px_{2n-1},Rx_{2n-1})}{d(Px_{2n},Px_{2n-1})}$$

$$+\gamma[d(Px_{2n},Qx_{2n})+d(Px_{2n-1},Rx_{2n-1})]+\delta[d(Px_{2n},Rx_{2n-1})+d(Px_{2n-1},Qx_{2n})]$$

$$\leq \alpha d(Px_{2n},Px_{2n-1}) + \frac{\beta d(Px_{2n},Px_{2n+1})d(Px_{2n-1},Px_{2n})}{d(Px_{2n},Px_{2n-1})}$$

$$+\gamma[d(Px_{2n},Px_{2n+1})+d(Px_{2n-1},Px_{2n})]+\delta[d(Px_{2n},Px_{2n})+d(Px_{2n-1},Px_{2n+1})]$$

$$d(Px_{2n+1},Px_{2n}) \leq h\, d(Px_{2n},Px_{2n-1}) \qquad \dots \qquad \textbf{(4)}$$

Where $h=\frac{(\alpha+\gamma+\delta)}{(1-\beta-\gamma-\delta)}$. Note that $h<1$, since $\alpha+\beta+2\gamma+2\delta<1$.

Similarly from (1) and (3), we get

$$d(Px_{2n-1,}Px_{2n}) = d(Qx_{2n-2}, Rx_{2n-1})$$
$$\leq \alpha d(Px_{2n-2}, Px_{2n-1}) + \frac{\beta d(Px_{2n-2}, Qx_{2n-2}) d(Px_{2n-1}, Rx_{2n-1})}{d(Px_{2n-2}, Px_{2n-1})}$$
$$+\gamma[d(Px_{2n-2}, Qx_{2n-2}) + d(Px_{2n-1}, Rx_{2n-1})]$$
$$+\delta[d(Px_{2n-2}, Rx_{2n-1}) + d(Px_{2n-1}, Qx_{2n-2})]$$
$$\leq \alpha d(Px_{2n-2}, Px_{2n-1}) + \frac{\beta d(Px_{2n-2}, Px_{2n-1}) d(Px_{2n-1}, Px_{2n})}{d(Px_{2n-2}, Px_{2n-1})}$$
$$+\gamma[d(Px_{2n-2}, Px_{2n-1}) + d(Px_{2n-1}, Px_{2n})]$$
$$+\delta[d(Px_{2n-2}, Px_{2n}) + d(Px_{2n-1}, Px_{2n-1})]$$
$$d(Px_{2n-1,}Px_{2n}) \leq h\, d(Px_{2n-2}, Px_{2n-1}) \quad \ldots \quad \textbf{(5)}$$

From (4) and (5), $d(Px_{2n+1,}Px_{2n}) \leq h^2 d(Px_{2n-2}, Px_{2n-1})$ for $n = 1,2,3, \ldots$, which by induction on n, implies that $d(Px_{2n+1,}Px_{2n}) \leq h^{2n}\, d(Px_1, Px_0)$ for all n. Letting $n \to \infty$, this gives $d(Px_{2n+1,}Px_{2n}) \to 0$, since $0 \leq h < 1$. Therefore, $\langle Px_n \rangle$is a Cauchy sequence in X. Since X is orbitally complete, we find a point $v \in X$ such that

$\lim_{n\to\infty} Px_{2n+1} = \lim_{n\to\infty} Qx_{2n} = \lim_{n\to\infty} Px_{2n+2} = \lim_{n\to\infty} Rx_{2n+1} = v$. **...(6)**

Now, the continuity of Q and P, the compatibility of the pair (Q, P) and (6) will give

$d(Qv, Pv) = \lim_{n\to\infty} d(QPx_{2n}, PQx_{2n}) = 0$ so that $Qv = Pv$. While the continuity of R and P, the compatibility of the pair (P, R) and (6) imply

$d(Rv, Pv) = \lim_{n\to\infty} d(RPx_{2n+1}, PRx_{2n+1}) = 0$ so that $Rv = Pv$. Hence

$Qv = Pv = Rv = z$, say. … **(7)**

Again from the continuity of Q and P, (6) and (7) we get

$\lim_{n\to\infty} d(Q^2x_{2n}, PQx_{2n}) = d(Qv, Pv) = 0$. Therefore in view of the compatibility of Q and P, this gives $d(PQv, QPv) = \lim_{n\to\infty} d(PQ^2x_{2n}, QPQx_{2n}) = 0$ or $PQv = QPv$.

Similarly from the continuity of R and P, (6), (7) and the compatibility of (R, P) we get $PRv = RPv$. Thus in view of (7), we get

$QQv = QPv = PQv = PRv = RPv = RQv$ … **(8)**

Now we prove z is a fixed point of R. If z is not a fixed point of R, then from (3), (7) and (8) we get

$$0 < d(z, Rz) = d(Qv, RQv) \leq \alpha d(Pv, PQv) + \frac{\beta d(Pv, Qv) d(PQv, RQv)}{d(Pv, PQv)}$$
$$+\gamma[d(Pv, Qv) + d(PQv, RQv)] + \delta[d(Pv, RQv) + d(PQv, Qv)]$$
$$d(z, Rz) \leq (\alpha + 2\delta) d(Pv, RQv) < d(Pv, RQv) = d(z, Rz).$$

Which is a contradiction. Thus we have Rz = z.

Hence from (7) and (8), we can say z is also a fixed point of Q and P. Thus z is a common fixed point of Q, R and P. The uniqueness of the fixed point can be easily proved.

Remark:

Since every complete metric space is orbitally complete at each of its points writing $Q = R$ and $P = I$, the identity map on X In the Inequality (3) of Theorem B we get the Theorem A.

References

[1] Gerald Jungck, Compatible mappings and common fixed points, *Int. Jour. Math. & Math. Sci*, 9 (1986), 771-779.

[2] Phaneendra T, Asymptotic regularity and common fixed point , Pure and App. Math. Sci., 59(1-2), (2004), 45-49.

[3] Sharma P.L. and Yuel A.K., A unique fixed point theorem in metric space, Bull. Cal. Math Soc., (76), (1984), 153-156.